

Differential Krull dimension in differential polynomial extensions

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Abstract

In the present paper, we investigate the differential Krull dimension of rings of differential polynomials. We prove a differential analogue of the Special Chain Theorem and show that some important classes of differential rings have no anomaly of differential Krull dimension.

Keywords: Differential polynomial rings, differential Krull dimension, chain conditions in differential polynomial rings

2010 MSC: 12H05, 13F05, 13C15

1. Introduction

In the theory of algebraic differential equations, we have the notion of the differential dimension of the set of solutions. Roughly speaking, the dimension shows how many independent solutions we have. But this characteristic describes the “global behavior”, that is the behavior of a “typical” solution. If we study solutions near a given one, many different anomalies occur. The examples can be found in [1, 2].

In [3, p. 607], it is noted that the notion of differential Krull dimension should play the desired role locally. And one of the most important questions is whether a differential integral domain being differentially finitely generated over a field is differentially catenary [3, p. 608]. This question is related with the behavior of descending chains of differential prime ideals. Such chains were investigated in [4] and [5]. But these papers did not provide a full answer. The last deep result about differential Krull dimension was obtained

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¹The work of the author was partially supported by the NSF Grant CCF-1016608.

in [5] and, after this paper, there were no results in this direction because of the lack of technique.

In the present paper, we investigate the differential Krull dimension of an extension by differentially transcendental elements in the ordinary case. Even this case has not been well-understood yet.

Precisely, we study possible bounds for the differential Krull dimension of a differential polynomial extension. The study was inspired by works of Jaffard and Seidenberg on the Krull dimension of polynomial extensions. It should be noted that it is more difficult to manipulate with differential Krull dimension in comparison with Krull dimension. In particular, this is because differential Krull dimension appears together with differential type and we have to deal with infinite sequences of differential prime ideals. Therefore, the classic methods used in the dimension theory of commutative rings do not work in the differential case.

Our main results are Theorem 29 and its applications (Theorems 30 and 43 and their corollaries). The theorem is a differential analogue of Jaffard's Special Chain Theorem [6, Théorème 3], which is a very important result on the behavior of prime chains in an arbitrary polynomial ring (for applications, see [7], [8], and [6]).

Rings with the well-behaved dimension of extensions are called Jaffard rings, that is rings of finite Krull dimension with $\dim R[x_1, \dots, x_n] = R + n$ for all n , i.e., the minimal possible value of the dimension holds. We call the corresponding class of differential rings, with the minimal possible dimension of extensions, as J-rings. As an application of Theorem 29, we obtained that all Jaffard rings are J-rings if they contain rational numbers or, more generally, are standard (Definition 7) (this condition is in some sense necessary, as Example 2 shows). Also, we proved that, under the same conditions, a differential ring with a locally nilpotent derivation is also a J-ring. This shows that both the commutative structure of a differential ring and properties of the derivation have an impact on the chains of differential prime ideals of differential polynomial extensions.

The paper is organized as follows. In Section 2, we give definitions and prove basic properties of differential Krull dimension. Section 3 is devoted to our main result, that is the Special Chain Theorem (Theorem 29). All technical lemmas are proved in Section 3.1. In Section 3.2, we prove Theorem 29 and its immediate corollaries. In Section 4, we introduce some classes of differential rings with “good” properties of differential Krull dimension. Section 4.1 deals with Δ -arithmetical rings; in Section 4.2, we study locally

nilpotent derivations, and, in Section 4.3, we apply the latter material and the Special Chain Theorem to well-known classes of rings.

2. Basic definitions and properties

All rings considered in this paper are assumed to be commutative and to contain an identity element. A differential ring is a ring with finitely many pairwise commuting derivations. An ordinary differential ring is a ring with one distinguished derivation. An ideal of a ring is called differential if, for each element of the ideal, its derivation belongs to the ideal. The set of all prime differential ideals of a ring R will be denoted by $\text{Spec}^\Delta R$ and will be called a differential spectrum.

A homomorphism $f: A \rightarrow B$, where A and B are differential rings, is called a differential homomorphism if it commutes with derivations. For an arbitrary differential ideal $\mathfrak{b} \subseteq B$, the ideal $f^{-1}(\mathfrak{b})$ is differential and is called the contraction of \mathfrak{b} . The contraction of \mathfrak{b} is denoted by \mathfrak{b}^c . For an arbitrary differential ideal $\mathfrak{a} \subseteq A$, the ideal $f(\mathfrak{a})B$ generated by the image of \mathfrak{a} is differential and is called the extension of \mathfrak{a} . The extension of \mathfrak{a} is denoted by \mathfrak{a}^e . If $\mathfrak{q} \subseteq B$ is a differential prime ideal, then its contraction is a prime differential ideal. Therefore, we have the map $f^*: \text{Spec}^\Delta B \rightarrow \text{Spec}^\Delta A$ by $\mathfrak{q} \mapsto f^{-1}(\mathfrak{q}) = \mathfrak{q}^c$. For a prime ideal $\mathfrak{p} \subseteq A$, the set $(f^*)^{-1}(\mathfrak{p})$ is called the fiber over \mathfrak{p} .

The definition of differential Krull dimension is given by Joseph Johnson in [5]. We will use a slightly modified version of the definition. Namely, we want the differential type of a differential ring with the only differential prime ideal to be zero.

Definition 1. Let R be a differential ring and $X \subset \text{Spec}^\Delta R$. There is a unique way to define the function called the “gap” measure

$$\mu: \{(\mathfrak{p}, \mathfrak{p}') \mid \mathfrak{p}, \mathfrak{p}' \in X, \mathfrak{p} \supset \mathfrak{p}'\} \rightarrow \mathbb{Z} \cup \infty,$$

such that the following conditions hold:

- 1) $\mu(\mathfrak{p}, \mathfrak{p}') \geq 0$
- 2) $\mu(\mathfrak{p}, \mathfrak{p}') = 0$ if and only if either $\mathfrak{p} = \mathfrak{p}'$, or there is no infinite descending chain of distinct differential primes in X such that $\mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \dots \supset \mathfrak{p}'$.

- 3) If $d > 0$ is a natural number, $\mu(\mathfrak{p}, \mathfrak{p}') \geq d$ if and only if $\mathfrak{p} \neq \mathfrak{p}'$ and there exists an infinite descending sequence $(\mathfrak{p}_i)_{i=0,1,\dots}$ of differential primes in X such that $\mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \dots \supset \mathfrak{p}'$ and $\mu(\mathfrak{p}_{i-1}, \mathfrak{p}_i) \geq d - 1$ for $i = 1, 2, \dots$

For an arbitrary subset $X \subseteq \text{Spec}^\Delta R$, define $\text{type}^\Delta X$ to be the least upper bound of all the $\mu(\mathfrak{p}, \mathfrak{p}')$, where $\mathfrak{p} \supseteq \mathfrak{p}'$ are elements of X , and call it the differential type of X . The differential type of a differential ring R is the differential type of its differential spectrum and is denoted by $\text{type}^\Delta R$. We define the differential dimension of X to be the least upper bound of $n \in \mathbb{Z}$ such that there exists a descending chain $\mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \dots \supset \mathfrak{p}_n$ of differential prime ideals in X with $\mu(\mathfrak{p}_{i-1}, \mathfrak{p}_i) = \text{type}^\Delta R$ for $i = 1, \dots, n$ and will denote it by $\dim^\Delta X$.

It should be noted that, in the definition of the differential dimension of a subset X , we use $\text{type}^\Delta R$ instead of $\text{type}^\Delta X$ for pairs of differential prime ideals. If $\text{Spec}^\Delta R$ is non-empty, $\text{type}^\Delta R$ is well-defined.

Definition 2. Let $\mathfrak{p}, \mathfrak{q} \in \text{Spec}^\Delta R$, $\mathfrak{p} \subset \mathfrak{q}$. Then we will use the following notation:

- The differential dimension of $\text{Spec}^\Delta R$ is called the differential dimension of R and is denoted by $\dim^\Delta R$.
- The differential height of \mathfrak{q} is the differential dimension of the set

$$\{\mathfrak{q}' \in \text{Spec}^\Delta R : \mathfrak{q}' \subset \mathfrak{q}\}$$

and is denoted by $\text{ht}^\Delta \mathfrak{q}$.

- The differential height of \mathfrak{q} over \mathfrak{p} is the differential dimension of the set

$$\{\mathfrak{q}' \in \text{Spec}^\Delta R : \mathfrak{p} \subset \mathfrak{q}' \subset \mathfrak{q}\}$$

and is denoted by $\text{ht}^\Delta \mathfrak{q}/\mathfrak{p}$.

- The differential type of \mathfrak{q} over \mathfrak{p} is the differential type of the set

$$\{\mathfrak{q}' \in \text{Spec}^\Delta R : \mathfrak{p} \subset \mathfrak{q}' \subset \mathfrak{q}\}$$

and is denoted by $\text{type}^\Delta \mathfrak{q}/\mathfrak{p}$.

Remark 3. It should be noted that, if $0 < \text{type}^\Delta R < \infty$, then $\dim^\Delta R > 0$.

If R is a differential ring and n is a positive integer, then $R^{\{n\}}$ will denote the differential polynomial ring $R\{z_1, \dots, z_n\}$ in n differential indeterminates over R and, if \mathfrak{a} is an ideal of R , then $\mathfrak{a}^{\{n\}}$ will denote the extension $\mathfrak{a}R^{\{n\}}$, that is the set of differential polynomials with coefficients in \mathfrak{a} .

Definition 4. Let \mathfrak{p} be a differential prime ideal of $R^{\{n\}}$ and \mathfrak{q} be a differential prime in R . We call \mathfrak{p} a differential upper to \mathfrak{q} if and only if $\mathfrak{p} \cap R = \mathfrak{q}$ and $\mathfrak{p} \neq \mathfrak{q}^{\{n\}}$. Otherwise, if $\mathfrak{p} = \mathfrak{q}^{\{n\}}$, then \mathfrak{p} is called an extended differential prime.

Definition 5. For every prime \mathfrak{p} in R , the field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ will be called the residue field of \mathfrak{p} and will be denoted by $k(\mathfrak{p})$.

The following theorem proved by Johnson in [5, Theorem of § 2] is the core result on differential Krull dimension that bears the rest of this section.

Theorem 6 (Johnson). *Let \mathbb{K} be a differential field of characteristic zero with m derivations. Then, for every $n \in \mathbb{N}$,*

$$\begin{aligned}\text{type}^\Delta \mathbb{K}^{\{n\}} &= m \\ \dim^\Delta \mathbb{K}^{\{n\}} &= n\end{aligned}$$

To study a fiber over a differential prime ideal via this theorem, we have to require that the residue field of the differential prime ideal has characteristic zero.

Definition 7. A differential ring R is called standard if, for every differential prime ideal \mathfrak{p} of R , $k(\mathfrak{p})$ has characteristic zero.

If R is standard, then, for every $n \geq 1$, $R\{z_1, \dots, z_n\}$ is also standard.

Corollary 8. *Let R be a standard ring with m derivations. Then, for every differential prime \mathfrak{p} of R , the fiber over \mathfrak{p} for the contraction map has differential type m and dimension n .*

Ritt algebras are common and the most important examples of standard rings. However, there are standard rings that are not Ritt algebras.

Example 1. Let R be the following ordinary differential ring

$$\mathbb{Z}_{(p)}\{x, y, z, u\}/[x^p - py, z^p - x_1 - pu, z_1 - 1].$$

We claim that R is not the zero ring and, for all prime differential ideals of R , their contractions to $\mathbb{Z}_{(p)}$ are zero.

To prove the first assertion, we consider $\mathbb{Q} \otimes_{\mathbb{Z}_{(p)}} R$. Hence, we have

$$\begin{aligned} \mathbb{Q} \otimes_{\mathbb{Z}_{(p)}} R &= \mathbb{Q}\{x, z, y, u\}/[z_1 - 1, y - x^p/p, u - (z^2 - x_1)/p] = \\ &= \mathbb{Q}\{x, z\}/[z_1 - 1] = \mathbb{Q}\{x\}[z]. \end{aligned}$$

In particular, we have just computed the fiber over the zero ideal, that coincides with the differential spectrum of the ring R .

To prove the second assertion, we consider $\mathbb{F}_p \otimes_{\mathbb{Z}_{(p)}} R$. Hence, we have

$$\begin{aligned} \mathbb{F}_p \otimes_{\mathbb{Z}_{(p)}} R &= \mathbb{F}_p\{x, y, u, z\}/[x^p, z^p - x_1, z_1 - 1] = \\ &= \mathbb{F}_p\{y, u\}[x, x_1, z, z_1]/(x^p, z^p - x_1, z_1 - 1) = \mathbb{F}_p\{y, u\}[z, x]/(x^p). \end{aligned}$$

Let J denote the smallest radical differential ideal of this ring. We will show that J is the whole ring. Indeed, since $x^p = 0$, $x \in J$. Since J is differential, $x_1 \in J$. From the equality $z^p = x_1$, it follows that $z \in J$. But $z_1 = 1$. So, $1 \in J$.

It should be noted that, if we are interested in the differential spectrum of a standard differential ring R , then we can suppose that R is a Ritt algebra. Indeed, we can replace R by the ring $\mathbb{Q} \otimes_{\mathbb{Z}} R$. The differential spectrum of the latter ring coincides with the differential spectrum of the ring R .

Since differential rings of characteristic p usually have big spectra, the assertion of Theorem 6 might be wrong, as the following example shows.

Example 2. Let $L = \mathbb{F}_p(x_n)_{n \in \mathbb{N}}$ be a purely transcendental extension of the finite field \mathbb{F}_p , the derivation ∂ is set to be zero. Then the ring of differential polynomials $L\{y\}$ has infinite differential type.

Indeed, for every subset $X \subseteq \mathbb{N}$, we define I_X to be the ideal generated by $\{y_k^p - x_k \mid k \in X\}$. The ideals I_X are differential prime ideals. In \mathbb{N} , one can find a descending sequence of subsets X_k such that $X_k \setminus X_{k+1}$ is countable. Thus, the family I_{X_k} is a descending chain of differential prime ideals. Since each set $X_k \setminus X_{k+1}$ is infinite, we can find a descending chain of sets $Y_{k,n}$

such that $Y_{k,n} \setminus Y_{k,n+1}$ is countable. Then we have a descending chain of the corresponding differential prime ideals $I_{Y_{k,n}}$.

Repeating this procedure for each $Y_{k,n}$, we can produce a chain of differential prime ideals of any differential type. Thus, the differential type of $L\{y\}$ is infinite.

Remark 9. Let S be a ring of finite Krull dimension, then we have the inequality ([9, Theorem 2])

$$\dim S + 1 \leq \dim S[x] \leq 2 \cdot \dim S + 1.$$

The next lemma shows possible bounds for the differential type of $R\{z\}$ in the spirit of the inequality in Remark 9.

Lemma 10. *Let R be a standard differential ring of differential type t with m derivations. Then we have*

$$\max(t, m) \leq \text{type}^\Delta R\{z_1, \dots, z_n\} \leq t + m.$$

Proof. Since every chain in R extends to a chain in $R\{z_1, \dots, z_n\}$, then $t \leq \text{type}^\Delta R^{\{n\}}$. Furthermore, there is a chain of differential type m by Corollary 8.

Let \mathfrak{C} be a chain realizing the differential type of $R^{\{n\}}$ and $\mathfrak{C} \cap R$ be the chain in R consisting of contractions of ideals of \mathfrak{C} . Since there is no chain of differential type $m + 1$ in the fiber over any differential prime, we have $\text{type}^\Delta(\mathfrak{C} \cap R) \geq \text{type}^\Delta R^{\{n\}} - m$. Therefore, $\text{type}^\Delta R \geq \text{type}^\Delta R^{\{n\}} - m$. \square

Corollary 11. *Let R be a standard differential rings of differential type t with m derivations and n is a positive integer. Then*

$$\text{type}^\Delta R^{\{n\}} \leq \text{type}^\Delta R^{\{n+1\}}.$$

Proof. Since $m \leq \text{type}^\Delta R^{\{n\}}$ by Lemma 10, then the inequality is clear after regarding $R^{\{n+1\}} = R^{\{n\}}\{z_{n+1}\}$. \square

Corollary 12. *Let R be a standard differential ring with m derivations. If $\mathfrak{p}, \mathfrak{q} \in \text{Spec}^\Delta R$, $\mathfrak{p} \subset \mathfrak{q}$ and $\text{type}^\Delta \mathfrak{q}/\mathfrak{p} = t$. Then, for any \mathfrak{q}' upper to \mathfrak{q} in $R^{\{n\}}$, we have $\text{type}^\Delta \mathfrak{q}'/\mathfrak{p}^{\{n\}} \leq t + m$.*

Proof. Since the ring $R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}$ has differential type t , the result follows from Lemma 10. \square

Lemma 13. *Let R be a standard differential ring of differential type t such that $\text{type}^\Delta R^{\{n\}} = \max(t, m)$. Then*

- a) $n \leq \dim^\Delta R^{\{n\}}$ if $t < m$;
- b) $n + \dim^\Delta R \leq \dim^\Delta R^{\{n\}}$ if $t = m$;
- c) $\dim^\Delta R \leq \dim^\Delta R^{\{n\}}$ if $t > m$.

Proof. In the case of $\dim^\Delta R = \infty$ the last two assertions are trivial, since every chain in R can be extended to $R^{\{n\}}$. The case of $t < m$ is straightforwardly deduced from Corollary 8.

Furthermore, let $\mathfrak{p}_0 \subset \dots \subset \mathfrak{p}_d$ be a chain realizing the differential dimension of R . Then, by Corollary 8, there is a chain $\mathfrak{p}_0^{\{n\}} \subset \dots \subset \mathfrak{p}_d^{\{n\}} \subset \mathfrak{q}_1 \subset \dots \subset \mathfrak{q}_n$, where all \mathfrak{q}_i are uppers to \mathfrak{p}_d . If $t = m$, this chain has dimension at least $d + n = \dim^\Delta R + n$. In the last case, the inequality holds since every chain in R can be extended to $R^{\{n\}}$ (the type of $\mathfrak{q}_1 \subset \dots \subset \mathfrak{q}_n$ less than $\text{type}^\Delta R^{\{n\}} = t$, so the chain has dimension zero). \square

There is no reason for dimension to be bounded above in general, but in special cases upper bound exists.

Lemma 14. *Let R be a standard differential ring of differential type t with m derivations. Then*

- 1. $n \leq \dim^\Delta R^{\{n\}} \leq n \cdot \dim^\Delta R$ if $t = 0$;
- 2. $1 \leq \dim^\Delta R^{\{n\}} \leq \dim^\Delta R$ if $\text{type}^\Delta R^{\{n\}} = t + m$.

Proof. (1). By Lemma 10, the type of $R^{\{n\}}$ is m . By Corollary 8, the fiber over every differential prime ideal has dimension n , therefore, $\dim^\Delta R^{\{n\}} \leq n \cdot \dim^\Delta R$. And the left-hand part follows Lemma 13.

(2). The left-hand inequality is obvious. Since the contraction of a chain of differential type $t + m$ in $R^{\{n\}}$ is a differential type t chain by Corollary 12, the right-hand inequality is clear. \square

Definition 15. Let R be a differential ring of finite differential type with m derivations. We say that R is a J-ring if, for every $n > 0$, $\max(t, m) = \text{type}^\Delta R\{z_1, \dots, z_n\}$ and

- 1. $\dim^\Delta R\{z_1, \dots, z_n\} = n$ if $t < m$;
- 2. $\dim^\Delta R < \infty$ and $\dim^\Delta R\{z_1, \dots, z_n\} = n + \dim^\Delta R$ if $t = m$;

3. $\dim^\Delta R < \infty$ and $\dim^\Delta R\{z_1, \dots, z_n\} = \dim^\Delta R$ if $t > m$.

These rings have the minimal possible values for both the differential type and the differential dimension of $R^{\{n\}}$, i.e., the lower bound of the inequalities in Lemma 13 holds. This is a differential analogue of Jaffard rings, that is, rings R with $\dim R[x_1, \dots, x_n] = \dim R + n$ for every n , i.e., the lower bound of the inequality in Remark 9 holds. The class of Jaffard rings is relatively wide, for example, finite dimensional Noetherian rings are Jaffard. Our goal is to establish similar results for the class of J-rings.

Lemma 16. *Let R be a differential ring of finite differential type t and \mathfrak{p} be a differential prime ideal in R . If $\text{ht}^\Delta \mathfrak{p} = \infty$, then, for every integer $n \geq 0$, there exists a differential prime $\mathfrak{p}_n \subset \mathfrak{p}$ such that $\text{ht}^\Delta \mathfrak{p}_n = n$.*

Proof. Suppose that the contrary holds. Obviously, if there is a differential prime ideal \mathfrak{q} with $\text{ht}^\Delta \mathfrak{q} = n$, then, for every $k \leq n$, there is a differential prime ideal \mathfrak{q} with $\text{ht}^\Delta \mathfrak{q} = k$. Now, we may suppose that there exists k such that our claim is false for k , then, for every differential prime $\mathfrak{q} \subset \mathfrak{p}$ such that $\text{ht}^\Delta \mathfrak{q} \geq k$, $\text{ht}^\Delta \mathfrak{q} = \infty$.

Let \mathfrak{q} be a differential prime of infinite height in R . We will find a differential prime \mathfrak{q}' in R such that $\text{type}^\Delta \mathfrak{q}/\mathfrak{q}' = t$ and $\text{ht}^\Delta \mathfrak{q}' = \infty$. Since $\text{ht}^\Delta \mathfrak{q} = \infty$, there is a chain

$$\mathfrak{p}^{k+1} \subset \dots \subset \mathfrak{p}^0 \subset \mathfrak{q}$$

such that $\text{type}^\Delta \mathfrak{p}^i/\mathfrak{p}^{i+1} = t$. Since $\text{ht}^\Delta \mathfrak{p}^1 \geq k$, $\text{ht}^\Delta \mathfrak{p}^1 = \infty$. Also $t = \text{type}^\Delta \mathfrak{p}^0/\mathfrak{p}^1 \leq \text{type}^\Delta \mathfrak{q}/\mathfrak{p}^1 \leq t$, therefore $\mathfrak{q}' = \mathfrak{p}^1$ is the required ideal.

Let $\mathfrak{q}_0 = \mathfrak{p}$. Applying the procedure above to \mathfrak{q}_i and denoting $\mathfrak{q}_{i+1} = \mathfrak{q}'_i$, we will obtain the chain $\mathfrak{p} = \mathfrak{q}_0 \supset \mathfrak{q}_1 \supset \mathfrak{q}_2 \supset \dots$ such that $\text{type}^\Delta \mathfrak{q}_i/\mathfrak{q}_{i+1} = t$. Hence, we have $\text{type}^\Delta R > t$, a contradiction. \square

The next two lemmas are quite similar, the only difference arises from the definition of J-rings. Namely, the case of differential rings with differential type less than the number of derivations is slightly different from another ones; the finiteness of the differential dimension is not assumed.

Lemma 17. *Let R be a differential ring of finite differential type $t < m$. If R is not a J-ring, then, for at least one pair of differential prime ideals $\mathfrak{p} \subset \mathfrak{m}$, $R_{\mathfrak{m}}/\mathfrak{p}R_{\mathfrak{m}}$ is not a J-ring.*

Proof. Suppose that the definition of a J-ring does not hold for a natural number $n > 0$. If $\text{type}^\Delta R^{\{n\}} > \max(t, m)$, then let $\mathfrak{q}_1 \subset \mathfrak{q}_2$ be differential primes in $R^{\{n\}}$ such that $\text{type}^\Delta \mathfrak{q}_2/\mathfrak{q}_1 = \text{type}^\Delta R^{\{n\}}$. Then $\mathfrak{q}_1 \cap R \subset \mathfrak{q}_2 \cap R$ is the required pair.

Now, suppose that $\text{type}^\Delta R^{\{n\}} = m = \max(t, m)$. Since $t < m$, we only need to show that the first condition of Definition 15 does not hold. Consider a chain $\mathfrak{q}_1 \subset \dots \subset \mathfrak{q}_r$ in $R^{\{n\}}$ such that $r = \dim^\Delta R^{\{n\}}$ if $\dim^\Delta R^{\{n\}} < \infty$, or $r = n + 1$ otherwise. In both cases, we have $r > n$. Then, obviously, our assertion holds for $\mathfrak{q}_1 \cap R \subset \mathfrak{q}_r \cap R$. \square

Lemma 18. *Let R be a differential ring of finite differential type $t \geq m$. Suppose that the ring R satisfies the following property: if $\text{type}^\Delta R^{\{n\}} = t$ for every n , then $\dim^\Delta R < \infty$. If R is not a J-ring, then, for at least one pair of differential prime ideals $\mathfrak{p} \subset \mathfrak{m}$, $R_{\mathfrak{m}}/\mathfrak{p}R_{\mathfrak{m}}$ is not a J-ring.*

Proof. Suppose that, for some natural number $n > 0$, $\text{type}^\Delta R^{\{n\}} > t = \max(t, m)$. Let $\mathfrak{q}_1 \subset \mathfrak{q}_2$ be differential primes in $R^{\{n\}}$ such that $\text{type}^\Delta \mathfrak{q}_2/\mathfrak{q}_1 = \text{type}^\Delta R^{\{n\}}$. Then $\mathfrak{q}_1 \cap R \subset \mathfrak{q}_2 \cap R$ is the required pair.

Now, suppose that $\text{type}^\Delta R^{\{n\}} = t$ for every n . Since $t \geq m$, we should contradict to the second and the third conditions of Definition 15. Under the hypothesis of the proposition, $\dim^\Delta R < \infty$. Let n be a natural number such that the definition of J-ring does not hold for n . There are two cases, either $\dim^\Delta R^{\{n\}}$ is finite or not. If $\dim^\Delta R^{\{n\}} = \infty$, there is a chain $\mathfrak{q}_1 \subset \dots \subset \mathfrak{q}_r$ with $r = \dim^\Delta R + n + 1$. Otherwise, we have either $t = m$ and $\dim^\Delta R^{\{n\}} > n + \dim^\Delta R$, or $t > m$ and $\dim^\Delta R^{\{n\}} > \dim^\Delta R$. In these cases, we consider a chain $\mathfrak{q}_1 \subset \dots \subset \mathfrak{q}_r$ in $R^{\{n\}}$ such that $r = \dim^\Delta R^{\{n\}}$. In all cases, we see that $\mathfrak{q}_1 \cap R \subset \mathfrak{q}_r \cap R$ is the desired pair of prime differential ideals of R . \square

Since we are mostly dealing with ordinary differential rings in the rest of the paper, the following lemma will be very useful.

Lemma 19. *Let \mathbb{K} be an ordinary differential field of characteristic zero. Then, for every nonzero differential prime ideal \mathfrak{p} of $\mathbb{K}\{z\}$,*

$$\text{ht}^\Delta \mathfrak{p} = 1.$$

Moreover, for every infinite descending chain C of prime differential ideals, we have

$$\bigcap_{\mathfrak{q} \in C} \mathfrak{q} = 0.$$

Proof. By Proposition 3 of [4, Chapter III, Section 2], we obtain the first assertion. Since the algebra $K\{z\}/\mathfrak{p}$ is differentially algebraic over K , the differential type of this quotient is 0 by Theorem in Section 2 of [5]. Hence, all descending chains of differential prime ideals of $K\{z\}/\mathfrak{p}$ are finite. \square

Remark 20. By Ritt-Raudenbush's basis theorem [10, Chapter II, Section 4, Theorem 1], all ascending chains in $\mathbb{K}^{\{n\}}$ have finite length if \mathbb{K} has characteristic zero, therefore, an infinite chain has to be descending.

3. Special Chain Theorem

In this section we prove our main result, that is Theorem 29. As an immediate corollary, we obtain that all ordinary standard Jaffard rings are J-rings.

3.1. Auxiliary lemmas

Lemma 21. *Let R be an ordinary standard differential ring with the condition $\text{type}^\Delta R\{z\} = 1$. Let $\mathfrak{q}' \subset \mathfrak{q}$ be differential primes in R such that $\text{type}^\Delta \mathfrak{q}/\mathfrak{q}' = 0$, $\mathfrak{p}' \subseteq \mathfrak{p}$ be differential primes in $R\{z\}$ of finite differential height such that $\mathfrak{p}' \cap R = \mathfrak{q}'$ and \mathfrak{p} is a differential upper to \mathfrak{q} . If $\text{ht}^\Delta \mathfrak{p} = \text{ht}^\Delta \mathfrak{p}' + 1$, then $\text{ht}^\Delta \mathfrak{p} = \text{ht}^\Delta \mathfrak{q}\{z\} + 1$.*

Proof. The inequality $\text{ht}^\Delta \mathfrak{p} \geq \text{ht}^\Delta \mathfrak{q}\{z\} + 1$ follows from Theorem 6.

Since there is a chain \mathfrak{C} between \mathfrak{p}' and \mathfrak{p} of differential type 1 and all descending chains between \mathfrak{q}' and \mathfrak{q} have a finite length, there is a differential prime \mathfrak{t} in R between \mathfrak{q}' and \mathfrak{q} such that there are infinite number of elements of \mathfrak{C} lying over \mathfrak{t} . By Remark 20, the chain over \mathfrak{t} has differential type 1, and, consequently, differential dimension 1, so $\text{ht}^\Delta \mathfrak{p}/\mathfrak{t}\{z\} = 1$. Therefore, $\text{ht}^\Delta \mathfrak{t}\{z\} \leq \text{ht}^\Delta \mathfrak{p} - 1$.

Since $\mathfrak{p}' \subset \mathfrak{p}_\alpha$ for each $\mathfrak{p}_\alpha \in \mathfrak{C}$ and

$$\mathfrak{t}\{z\} = \bigcap_{\substack{\mathfrak{p}_\alpha \in \mathfrak{C} \\ \mathfrak{p}_\alpha \cap R = \mathfrak{t}}} \mathfrak{p}_\alpha$$

by Lemma 19, we have $\mathfrak{p}' \subset \mathfrak{t}\{z\}$ and, therefore, $\text{ht}^\Delta \mathfrak{p}' \leq \text{ht}^\Delta \mathfrak{t}\{z\}$.

So, we have

$$\text{ht}^\Delta \mathfrak{p} = \text{ht}^\Delta \mathfrak{p}' + 1 \leq \text{ht}^\Delta \mathfrak{t}\{z\} + 1 \leq \text{ht}^\Delta \mathfrak{q}\{z\} + 1.$$

\square

Lemma 22. *Let R be an ordinary standard differential ring of finite differential type, $n \geq 1$ be an integer such that $\text{type}^\Delta R^{\{n\}} = \text{type}^\Delta R$, and \mathfrak{p} be a differential prime ideal in $R^{\{n\}}$. If $\text{ht}^\Delta \mathfrak{p} < \infty$, then $\text{ht}^\Delta(\mathfrak{p} \cap R) < \infty$.*

Proof. Since every chain of differential prime ideals in R can be extended to a chain in $R^{\{n\}}$ of the same differential type, the differential height of the contraction of \mathfrak{p} cannot be infinite. \square

Corollary 23. *Let R be a standard differential ring with one derivation such that $\text{type}^\Delta R \leq 1$. Let $n \geq 1$ be an integer such that $\text{type}^\Delta R^{\{n\}} = 1$. If a differential prime \mathfrak{p} in $R^{\{n\}}$ has finite differential height, then, for every $0 < k < n$, $\mathfrak{p} \cap R^{\{k\}}$ also has finite differential height.*

Proof. By Corollary 11, we have $\text{type}^\Delta R^{\{k\}} = 1$ for $k > 0$. Therefore, the result follows from Lemma 22. \square

Lemma 24. *Let R be an ordinary standard differential ring such that*

$$\text{type}^\Delta R \leq 1 \text{ and } \text{type}^\Delta R\{z\} = 1.$$

Let \mathfrak{q} be a differential prime ideal in R and \mathfrak{p} be a differential upper to \mathfrak{q} in $R\{z\}$. If $\text{ht}^\Delta \mathfrak{p} < \infty$, then

$$\text{ht}^\Delta \mathfrak{p} = \text{ht}^\Delta \mathfrak{q}\{z\} + 1.$$

If we additionally suppose that, for some natural number $n > 0$, $\text{type}^\Delta R^{\{n\}} = 1$ and $\mathfrak{p}\{z_2, \dots, z_n\}$ has finite differential height, then

$$\text{ht}^\Delta \mathfrak{p}\{z_2, \dots, z_n\} = \text{ht}^\Delta \mathfrak{q}\{z_1, \dots, z_n\} + 1.$$

Proof. The inequality $\text{ht}^\Delta \mathfrak{p} \geq \text{ht}^\Delta \mathfrak{q}\{z\} + 1$ follows from Lemma 19. So, we have to show the inverse one.

Since $\text{ht}^\Delta \mathfrak{p}/\mathfrak{q}\{z\} = 1$, then $m = \text{ht}^\Delta \mathfrak{p} \geq 1$. Let \mathfrak{p}_1 be a differential prime with $\text{ht}^\Delta \mathfrak{p}_1 = m - 1$ in a chain realizing differential height of \mathfrak{p} and $\mathfrak{q}_1 = \mathfrak{p}_1 \cap R$. If $\text{type}^\Delta R = 0$, then $\text{type}^\Delta \mathfrak{q}/\mathfrak{q}_1 = 0$ and we are done by Lemma 21. So, we may suppose that $\text{type}^\Delta R = 1$.

We use the induction by $\text{ht}^\Delta \mathfrak{q}$. If $\text{ht}^\Delta \mathfrak{q} = 0$, then $\text{type}^\Delta \mathfrak{q}/\mathfrak{q}_1 = 0$ and we are done by Lemma 21.

If $\text{ht}^\Delta \mathfrak{q}/\mathfrak{q}_1 \geq 1$, then

$$\text{ht}^\Delta \mathfrak{p} = \text{ht}^\Delta \mathfrak{p}_1 + 1 \leq \text{ht}^\Delta \mathfrak{q}_1\{z\} + 2 \leq \text{ht}^\Delta \mathfrak{q}\{z\} + 1.$$

As for the second assertion, we have

$$\mathfrak{p}\{z_2, \dots, z_n\} \cap R\{z_2, \dots, z_n\} = \mathfrak{q}\{z_2, \dots, z_n\}$$

and, after considering the ring $R\{z_2, \dots, z_n\}$ as R , $\mathfrak{p}\{z_2, \dots, z_n\}$ as \mathfrak{p} , and $\mathfrak{q}\{z_2, \dots, z_n\}$ as \mathfrak{q} , the result follows from the first part of the lemma. \square

Proposition 25. *Let R be an ordinary standard differential ring such that $\text{type}^\Delta R \leq 1$, $n \geq 1$ be an integer number such that $\text{type}^\Delta R^{\{n\}} = 1$, and \mathfrak{p} be a differential upper to \mathfrak{q} in $R^{\{n\}}$. If \mathfrak{p} has a finite differential height, then*

$$\text{ht}^\Delta \mathfrak{p} = \text{ht}^\Delta \mathfrak{q}^{\{n\}} + \text{ht}^\Delta \mathfrak{p}/\mathfrak{q}^{\{n\}}.$$

Proof. The inequality $\text{ht}^\Delta \mathfrak{p} \geq \text{ht}^\Delta \mathfrak{q}^{\{n\}} + \text{ht}^\Delta \mathfrak{p}/\mathfrak{q}^{\{n\}}$ is clear. So, we must show the inverse one.

We induce by n . The case $n = 1$ immediately follows from Lemma 24. Therefore, we assume that the result holds for all $k < n$. Set $\mathfrak{p}_1 = \mathfrak{p} \cap R\{z_1\}$, then \mathfrak{p}_1 has a finite differential height by Corollary 23. If $\mathfrak{p}_1 = \mathfrak{q}\{z_1\}$, then, representing $R\{z_1, \dots, z_n\}$ as $R\{z_1\}\{z_2, \dots, z_n\}$, we derive the result from the induction hypothesis.

If $\mathfrak{q}\{z_1\} \subset \mathfrak{p}_1$, then Lemma 24 implies that

$$\text{ht}^\Delta \mathfrak{p}_1\{z_2, \dots, z_n\} = \text{ht}^\Delta \mathfrak{q}\{z_1, \dots, z_n\} + 1$$

and if $\mathfrak{p} = \mathfrak{p}_1\{z_2, \dots, z_n\}$, then we are done. Now, we may suppose that $\mathfrak{p} \supset \mathfrak{p}_1\{z_2, \dots, z_n\}$. Therefore, the induction assumption implies that

$$\begin{aligned} \text{ht}^\Delta \mathfrak{p} &= \text{ht}^\Delta \mathfrak{p}_1\{z_2, \dots, z_n\} + \text{ht}^\Delta \mathfrak{p}/\mathfrak{p}_1\{z_2, \dots, z_n\} = \\ &= 1 + \text{ht}^\Delta \mathfrak{q}\{z_1, \dots, z_n\} + \text{ht}^\Delta \mathfrak{p}/\mathfrak{p}_1\{z_2, \dots, z_n\}. \end{aligned}$$

Since $\mathfrak{q}\{z_1\} \subset \mathfrak{p}_1$, $\text{ht}^\Delta \mathfrak{p}_1/\mathfrak{q}\{z_1\} = 1$ by Lemma 19. Also, since every differential prime \mathfrak{q}' in $R\{z_1\}$ can be extended to the differential prime $\mathfrak{q}'\{z_2, \dots, z_n\}$ in $R^{\{n\}}$, every chain between $\mathfrak{q}\{z_1\}$ and \mathfrak{p}_1 can be extended to a chain between $\mathfrak{q}\{z_1, \dots, z_n\}$ and $\mathfrak{p}_1\{z_2, \dots, z_n\}$. Therefore, we have

$$1 + \text{ht}^\Delta \mathfrak{p}/\mathfrak{p}_1\{z_2, \dots, z_n\} \leq \text{ht}^\Delta \mathfrak{p}/\mathfrak{q}\{z_1, \dots, z_n\}.$$

The latter inequality implies that

$$\text{ht}^\Delta \mathfrak{p} \leq \text{ht}^\Delta \mathfrak{q}\{z_1, \dots, z_n\} + \text{ht}^\Delta \mathfrak{p}/\mathfrak{q}\{z_1, \dots, z_n\}.$$

\square

Lemma 26. *Let R be an ordinary standard differential ring satisfying the condition $\text{type}^\Delta R^{\{n\}} = t \geq 2$. Let $\mathfrak{q}' \subset \mathfrak{q}$ be differential primes in R such that $\text{type}^\Delta \mathfrak{q}/\mathfrak{q}' < t$, $\mathfrak{p}' \subseteq \mathfrak{p}$ be differential primes in $R^{\{n\}}$ of finite differential height such that $\mathfrak{p}' \cap R = \mathfrak{q}'$ and \mathfrak{p} is a differential upper to \mathfrak{q} . If $\text{ht}^\Delta \mathfrak{p} = \text{ht}^\Delta \mathfrak{p}' + 1$, then $\text{ht}^\Delta \mathfrak{p} = \text{ht}^\Delta \mathfrak{q}^{\{n\}}$.*

Proof. The inequality $\text{ht}^\Delta \mathfrak{p} \geq \text{ht}^\Delta \mathfrak{q}^{\{n\}}$ is clear, we must show the inverse one.

By Corollary 12 and the assumption on $\text{type}^\Delta \mathfrak{q}'/\mathfrak{q}$, we have $\text{type}^\Delta \mathfrak{q}/\mathfrak{q}' = t - 1$. There is the only way to get a chain of differential type t between \mathfrak{p} and \mathfrak{p}' . Namely, there is a chain $C = \{\mathfrak{q}' \subset \dots \subset \mathfrak{q}_\alpha \subset \dots \subset \mathfrak{q}_0 \subset \mathfrak{q}\}$ such that, for every $\mathfrak{q}_\alpha \in C$, there exists \mathfrak{p}_α being a differential upper to \mathfrak{q}_α such that $\mathfrak{p}_\alpha \subset \mathfrak{q}_{\alpha-1}^{\{n\}}$ for every α . Therefore, $\text{ht}^\Delta \mathfrak{p} = \text{ht}^\Delta \mathfrak{q}_0^{\{n\}} \leq \text{ht}^\Delta \mathfrak{q}^{\{n\}}$. \square

Lemma 27. *Let R be an ordinary standard differential ring with the condition $\text{type}^\Delta R = t > 1$. Let \mathfrak{q} be a differential prime in R and \mathfrak{p} be an upper to \mathfrak{q} in $R\{z\}$ of finite differential height. Then*

$$\text{ht}^\Delta \mathfrak{p} = \text{ht}^\Delta \mathfrak{q}\{z\},$$

and, for any positive integer n such that $\text{ht}^\Delta \mathfrak{p}\{z_2, \dots, z_n\} < \infty$, we have

$$\text{ht}^\Delta \mathfrak{p}\{z_2, \dots, z_n\} = \text{ht}^\Delta \mathfrak{q}^{\{n\}}.$$

Proof. The inequality $\text{ht}^\Delta \mathfrak{p} \geq \text{ht}^\Delta \mathfrak{q}\{z\}$ is clear. If $\text{ht}^\Delta \mathfrak{p} = 0$, we are done. So, we may suppose that $\text{ht}^\Delta \mathfrak{p} \geq 1$. Let \mathfrak{p}' be a differential prime in a chain realizing the differential height of \mathfrak{p} with $\text{ht}^\Delta \mathfrak{p}' = \text{ht}^\Delta \mathfrak{p} - 1$ and let $\mathfrak{q}' = \mathfrak{p}' \cap R$. From Corollary 12, it follows that

$$\text{type}^\Delta R\{z\} - 1 \leq \text{type}^\Delta \mathfrak{q}/\mathfrak{q}' \leq \text{type}^\Delta R \leq \text{type}^\Delta R\{z\}.$$

The case of $\text{type}^\Delta \mathfrak{q}/\mathfrak{q}' = \text{type}^\Delta R\{z\} - 1$ follows from Lemma 26.

In the second case, by Lemma 22, we have that $\text{ht}^\Delta \mathfrak{q} < \infty$ and $\text{ht}^\Delta \mathfrak{q}' < \text{ht}^\Delta \mathfrak{q}$. We will use the induction by $\text{ht}^\Delta \mathfrak{q}$. In the case of $\text{ht}^\Delta \mathfrak{q} = 1$, we have $\text{ht}^\Delta \mathfrak{p}' = \text{ht}^\Delta \mathfrak{q}'\{z\}$ by the first part of the proof. Otherwise, by the induction hypothesis, we have $\text{ht}^\Delta \mathfrak{p}' = \text{ht}^\Delta \mathfrak{q}'\{z\}$. Therefore,

$$\text{ht}^\Delta \mathfrak{p} = \text{ht}^\Delta \mathfrak{p}' + 1 = \text{ht}^\Delta \mathfrak{q}'\{z\} + 1 \leq \text{ht}^\Delta \mathfrak{q}\{z\}.$$

Since $\mathfrak{q}\{z_2, \dots, z_n\} = R\{z_2, \dots, z_n\} \cap \mathfrak{p}\{z_2, \dots, z_n\}$, then, after considering $R^{\{n\}}$ as $R\{z_2, \dots, z_n\}\{z_1\}$, the second assertion follows from the first one. \square

Lemma 28. *Let R be an ordinary standard differential ring satisfying the condition $\text{type}^\Delta R = t \geq 1$. Let \mathfrak{p} be a differential prime of finite differential height in $R^{\{n\}}$ and $\mathfrak{p} \cap R = \mathfrak{q}$. Then, for every positive integer n such that $\text{type}^\Delta R^{\{n\}} \geq 2$, we have $\text{ht}^\Delta \mathfrak{p} = \text{ht}^\Delta \mathfrak{q}^{\{n\}}$.*

Proof. If $\mathfrak{p} = \mathfrak{q}^{\{n\}}$ our claim is trivial.

The case of $\text{type}^\Delta R = 1$ follows from Lemma 26 since $\text{type}^\Delta R^{\{n\}} \geq 2$. We may suppose that $\text{type}^\Delta R > 1$, then, by Lemma 10, $\text{type}^\Delta R^{\{n\}} \geq \text{type}^\Delta R > 1$ for every n .

We use the induction by n . The base follows from Lemma 27. Therefore, we assume that the result holds for all $k < n$. Set $\mathfrak{p}_1 = \mathfrak{p} \cap R^{\{n-1\}}$. By Lemma 27, $\text{ht}^\Delta \mathfrak{p}_1\{z_n\} = \text{ht}^\Delta \mathfrak{p}$, so $\mathfrak{p}_1\{z_n\}$ has finite differential height. If $\mathfrak{p}_1\{z_n\} = \mathfrak{q}^{\{n\}}$, then we are done. Otherwise, $\mathfrak{p}_1\{z_n\}$ is an upper to $\mathfrak{q}\{z_n\}$ because $\mathfrak{p}_1\{z_n\} \cap R\{z_n\} = \mathfrak{q}\{z_n\}$. After regarding $R^{\{n\}}$ as $R\{z_n\}^{\{n-1\}}$, we have $\text{ht}^\Delta \mathfrak{p}_1\{z_n\} = \text{ht}^\Delta \mathfrak{q}^{\{n\}}$ by the induction hypothesis. \square

3.2. The main theorem and its corollaries

Theorem 29 (Special Chain Theorem). *Let R be an ordinary standard differential ring of finite differential type, then, for every positive integer n , the following holds*

$$\text{ht}^\Delta \mathfrak{p} = \text{ht}^\Delta \mathfrak{q}^{\{n\}} + \text{ht}^\Delta \mathfrak{p}/\mathfrak{q}^{\{n\}},$$

where $\mathfrak{p} \in \text{Spec}^\Delta R^{\{n\}}$ and $\mathfrak{p} \cap R = \mathfrak{q}$.

Proof. Let $\text{ht}^\Delta \mathfrak{p} < \infty$. If $\text{type}^\Delta R^{\{n\}} = 1$, the result follows from Proposition 25, otherwise from Lemma 28.

If $\text{ht}^\Delta \mathfrak{p} = \infty$, then it follows from Lemma 16 that, for every i , there is $\mathfrak{p}_i \subset \mathfrak{p}$ with $\text{ht}^\Delta \mathfrak{p}_i = i$. By the case of finite differential height, we have $\text{ht}^\Delta (\mathfrak{p}_i \cap R)^{\{n\}} \geq i - n$. Therefore, $\text{ht}^\Delta \mathfrak{q}^{\{n\}} = \infty$. \square

This differential analogue of Jaffard's theorem (see [7, 2.3] and [8, Theorem 1]) will be our main tool for the investigation of the differential dimension of certain classes of rings.

Jaffard proved in [6, Théorème 2] that the sequence

$$d_n = \dim R[x_1, \dots, x_n] - \dim R[x_1, \dots, x_{n-1}]$$

is eventually constant and the eventual value d_R is at most $\dim R + 1$. Therefore, there is an equality $\dim R[x_1, \dots, x_n] = d_R \cdot n + C$ for some constant C and sufficiently large n -th.

For each ideal $\mathfrak{q} \subset R^{\{n\}}$, we use the following notation

$$\mathfrak{q}_m = \mathfrak{q} \cap R[z_1, \dots, \partial^m z_1, \dots, z_n, \dots, \partial^m z_n].$$

Let R be an ordinary standard differential ring of finite Krull dimension, then $\text{type}^\Delta R = 0$ and, by Lemma 10, $\text{type}^\Delta R^{\{n\}} = 1$ for every n . Also, by Lemma 14, $\dim^\Delta R^{\{n\}} < \infty$.

Theorem 30. *Let R be an ordinary standard differential ring of finite Krull dimension. If, for some n , we have $\dim^\Delta R^{\{n\}} > n$, then $d_R > 1$.*

Proof. Let \mathfrak{m} be a differential prime in $R^{\{n\}}$ and $\mathfrak{q} = \mathfrak{m} \cap R$ such that $\text{ht}^\Delta \mathfrak{m} = \dim^\Delta R^{\{n\}} > n$. By Theorem 29, we have $\text{ht}^\Delta \mathfrak{m} = \text{ht}^\Delta \mathfrak{q}^{\{n\}} + \text{ht}^\Delta \mathfrak{m}/\mathfrak{q}^{\{n\}}$. Also, by Corollary 8, $\text{ht}^\Delta \mathfrak{m}/\mathfrak{q}^{\{n\}} \leq n$. Therefore, $\text{ht}^\Delta \mathfrak{q}^{\{n\}} > 0$ and, by the argument similar to the proof of Lemma 21, there are a differential prime $\mathfrak{t} \subset \mathfrak{q}$ in R and a differential prime ideal \mathfrak{p} in $R^{\{n\}}$ that is upper to \mathfrak{t} such that $\mathfrak{p} \subset \mathfrak{q}^{\{n\}}$. Since we have the inclusion

$$\mathfrak{p}_m \subset \mathfrak{q}[z_1, \dots, \partial^m z_1, \dots, z_n, \dots, \partial^m z_n]$$

and there is a chain of $n \cdot (m+1)$ primes in $R[z_1, \dots, \partial^m z_1, \dots, z_n, \dots, \partial^m z_n]$ over \mathfrak{q} , then $\text{ht} \mathfrak{p}_m + n \cdot (m+1) < d_R \cdot n \cdot (m+1) + C$. Assume that $d_R = 1$, then we have $\text{ht} \mathfrak{p}_m < C$.

There is an infinite chain $\mathfrak{p} = \mathfrak{p}^0 \supset \mathfrak{p}^1 \supset \dots \supset \mathfrak{t}^{\{n\}}$ of differential type 1. Let f_i be polynomials such that $f_i \in \mathfrak{p}^{i-1}$ and $f_i \notin \mathfrak{p}^i$. Suppose k_j is the minimal integer such that

$$f_i \in R[z_1, \dots, \partial^{k_j} z_1, \dots, z_n, \dots, \partial^{k_j} z_n] \text{ for all } i < j.$$

Therefore, there is a chain $\mathfrak{p}_{k_j} \supset \mathfrak{p}_{k_j}^1 \supset \dots \supset \mathfrak{p}_{k_j}^j$ and

$$\text{ht} \mathfrak{p}_{k_j} > j.$$

A contradiction with $d_R = 1$. □

Corollary 31. *If R is a Jaffard ring that is ordinary and standard, then R is a J-ring.*

Proof. By definition, R has finite Krull dimension and $d_n = 1$ for every n . □

Corollary 32. *If R is a Noetherian standard differential ring with one derivation, then R is a J-ring.*

Proof. Since a Noetherian local ring has finite Krull dimension ([11, Corollary 11.11]), all chains in R have finite length. Hence, $\text{type}^\Delta R = 0$.

Supoe that our assumption is wrong. Since $\text{type}^\Delta R = 0$ and R is ordinary, by Lemma 17, $R_{\mathfrak{m}}$ is not a J-ring for some \mathfrak{m} . But $R_{\mathfrak{m}}$ is local Noetherian and, hence, Jaffard by [8, Corollary 2]. \square

Remark 33. In the case of $\text{type}^\Delta R < \text{type}^\Delta R^{\{n\}}$, one should not expect a “good” behavior of differential height. For example, let R be the Nagata example [12, A1, p. 203, Example 1] of a Noetherian ring with $\dim R = \infty$ and ∂ be the zero derivation. We have $\text{type}^\Delta R = 0$, and there are prime ideals such that $\text{ht}^\Delta \mathfrak{q} = k$ for every k . However, for every prime ideal \mathfrak{q} in R , $\text{ht}^\Delta \mathfrak{q}\{z\} = 0$. Furthermore, we expect that there exists a ring with $\text{ht}^\Delta \mathfrak{p} = \infty$ and $\text{ht}^\Delta \mathfrak{p}\{z\} = 0$.

4. Examples of J-rings

In this section we are going to establish some well-known classes of rings included in the class of J-rings.

4.1. Δ -arithmetical rings

Our first class is an analogue of arithmetical rings. Arithmetical rings are important since every Prüfer domain is arithmetical.

Definition 34. A ring A is called arithmetical if every finitely generated ideal is locally principal, that is, for every finitely generated ideal \mathfrak{a} , the image of \mathfrak{a} in $A_{\mathfrak{p}}$ is principal for every prime ideal \mathfrak{p} in A .

Arithmetical rings are Jaffard [13, Theorem 4].

Definition 35. A Δ -arithmetical ring is a differential ring A such that, for every finitely generated ideal \mathfrak{a} , $\mathfrak{a}A_{\mathfrak{p}}$ is principle ideal for every differential prime \mathfrak{p} .

Remark 36. It is well-known that an integer domain is arithmetical if and only if it is a Prüfer domain [14, Chapter 1, Section 6, Theorem 62]. Consider a local Δ -arithmetical domain, such that the maximal ideal is differential. Then this ring is a local Prüfer domain, i.e. a valuation domain.

The following example shows that there are Δ -arithmetical rings that are not arithmetical.

Example 3. Let O be the Krull example of an integrally closed one-dimensional local domain not being a valuation ring [15, p. 670f]. We will recall the construction. Let K be an algebraically closed field, x and y are indeterminates. The ring O consists of the rational functions $r(x, y)$ such that x does not divide denominator of $r(x, y)$ and $r(0, y) \in K$. Let y and elements of K be constant and $x' = 1$.

The ring O is local and is not a valuation ring, so, O is not a Prüfer domain. The only differential prime is (0) and, in the field of fractions of O , every ideal is principle.

Proposition 37. *Let \mathfrak{p} be a differential prime ideal of a Δ -arithmetical domain O . If \mathfrak{q} is a differential upper to zero, then $\mathfrak{q} \not\subseteq \mathfrak{p}\{z_1, \dots, z_n\}$.*

Proof. We will follow Seidenberg's proof ([9, Theorem 4]).

We may assume that O is local and \mathfrak{p} is the maximal ideal. Then O is a valuation domain. Let $f \in \mathfrak{q}$, then $f = c \cdot g$, where $c \in \mathfrak{p}$ and g has at least one coefficient equal to 1. Therefore, $g \in \mathfrak{q}$ and $g \notin \mathfrak{p}\{z_1, \dots, z_n\}$. \square

4.2. Derivations with special properties

Let us recall that the characteristic of a ring R is the natural number n such that $n\mathbb{Z}$ is the kernel of the unique ring homomorphism from \mathbb{Z} to R .

Lemma 38. *Let O be a differential integral domain of characteristic zero, \mathfrak{p} is a differential upper to zero in $O\{z_1, \dots, z_n\}$. If there is a nonzero polynomial $f \in \mathfrak{p}$ such that its coefficients are constant for some $\partial \in \Delta$, then there is $f^* \in \mathfrak{p}$ such that f^* has relatively prime coefficients in \mathbb{Z} .*

Proof. Every non-zero polynomial f with constant coefficients can be presented as the following sum

$$a_1 f_1 + \dots + a_n f_n,$$

where $a_i \in O$ are constant and f_i are polynomials with integer coefficients. Let $f \in \mathfrak{p}$ be taken such that n is minimal among all such numbers. If $n = 1$, then $f = a f_1 \in \mathfrak{p}$. Since \mathfrak{p} is prime and $\mathfrak{p} \cap O = 0$, $f_1 \in \mathfrak{p}$.

Suppose that $n > 1$, then

$$\partial f = a_1 \partial f_1 + \dots + a_n \partial f_n.$$

Set g to be the following polynomial

$$f_1 \partial f - \partial f_1 f = a_2 g_2 + \dots + a_n g_n,$$

where $g_i = \partial f_i f_1 - \partial f_1 f_i$. The polynomial g has constant coefficients. From the definition of f , it follows that $g = 0$. Let L be the field of fractions of O , then the fraction f_1/f is a ∂ -constant in $L\langle y_1, \dots, y_n \rangle$. Therefore, f_1/f belongs to the subfield of ∂ -constants of L [10, Chapter II, Section 9, Corollary 5 of Theorem 4]. In particular, $af_1 = bf$ for some elements $a, b \in O$. Thus, $af_1 \in \mathfrak{p}$ and again $f_1 \in \mathfrak{p}$.

So, we have a polynomial f with integral coefficients in \mathfrak{p} . We may suppose that coefficients of f are coprime because $f = df'$, where d is the greatest common divisor of the coefficients. Since \mathfrak{p} is an upper to zero, $d \notin \mathfrak{p}$. Thus, $f' \in \mathfrak{p}$. \square

Remark 39. If the ring O is a Ritt algebra in the previous theorem, then all coefficients of f^* are invertible.

Lemma 40. *Let (O, \mathfrak{m}) be a differential local integral domain of characteristic zero and \mathfrak{p} be an upper to zero in $\text{Spec}^\Delta O\{z_1, \dots, z_n\}$. If there is $f \in \mathfrak{p}$ with the property: there is $\partial \in \Delta$ such that every coefficient a of f satisfies $\partial^n(a) = 0$ for some natural n . Then there exists $h \in \mathfrak{p}$ having relatively prime coefficients in \mathbb{Z} .*

Proof. We will say that $a \in O$ is ∂ -nilpotent if $\partial^n(a) = 0$ for some n . If $a \in O$ is ∂ -nilpotent then we define $d(a)$ to be the maximal $n \in \mathbb{N}$ such that $\partial^n(a) \neq 0$.

Every polynomial f with ∂ -nilpotent coefficients can be presented as the following sum

$$a_1 f_1 + \dots + a_n f_n,$$

where $a_i \in O$ are ∂ -nilpotent, $d(a_1) \geq d(a_2) \geq \dots \geq d(a_n)$, and all f_i have integer coefficients. For such representation of f , we set $d(f) = d(a_1)$ and define $n(f)$ to be the maximal number i such that $d(a_i) = d(f)$.

Let f be a non-zero polynomial of \mathfrak{p} with ∂ -nilpotent coefficients and its representation

$$a_1 f_1 + \dots + a_n f_n$$

be chosen such that the pair $(d(f), n(f))$ is lexicographically minimal. If $d(f) = 0$, then the lemma follows from Lemma 38. Suppose that $d(f) > 0$. Let g be as follows

$$g = f_1 \partial f - \partial f_1 f = a_2 g_2 + \dots + a_n g_n + \partial(a_1) h_1 + \dots + \partial(a_n) h_n,$$

where $g_i = \partial f_i f_1 - \partial f_1 f_i$, $h_i = f_1 f_i$. Then g has ∂ -nilpotent coefficients and this representation for g is less than the initial representation of f . Thus, $g = 0$. So, we have $f_1 \partial f = \partial f_1 f$. As in the proof of Lemma 38, we derive that $a f_1 \in \mathfrak{p}$ for some $a \in O$. And since $\mathfrak{p} \cap O = 0$, $f_1 \in \mathfrak{p}$. \square

Lemma 41. *Let (O, \mathfrak{m}) be a differential local integral domain of characteristic zero, \mathfrak{p} is a differential upper to zero in $O\{z_1, \dots, z_n\}$. If there is $f \in \mathfrak{p}$ such that all coefficients are ∂ -nilpotent for some $\partial \in \Delta$, then $\mathfrak{p} \not\subseteq \mathfrak{m}\{z_1, \dots, z_n\}$.*

Proof. By Lemma 40, there is $f^* \in \mathfrak{p}$ with coefficients in \mathbb{Z} . If $\mathfrak{m} \cap \mathbb{Z} = 0$ then $O \supset \mathbb{Q}$ and, by Remark 39, $f^* \notin \mathfrak{m}^{\{n\}}$. Otherwise, since O is local, $\mathbb{Z}_{(p)} \subset O$ for some prime p . Therefore, since coefficients are relatively prime, at least one coefficient is not divisible by p . Thus, $f^* \notin \mathfrak{m}^{\{n\}}$. \square

4.3. An application of the Special Chain Theorem

We will say that a differential ring R satisfies the property **S3**, if, for every natural number $n > 0$, for every differential prime ideals $\mathfrak{q}_1 \subset \mathfrak{q}_2$ in R , there is no differential prime \mathfrak{p} in $R^{\{n\}}$ such that \mathfrak{p} is an upper to \mathfrak{q}_1 and $\mathfrak{p} \subset \mathfrak{q}_2^{\{n\}}$. This property has a sense like being a Stable Strong S-Domain for usual polynomial extensions.

Remark 42. If **S3** holds, every element of a chain that ends in $\mathfrak{q}^{\{n\}}$ is an extended prime too. Therefore, for every chain \mathfrak{C} that ends in extended prime, $\text{type}^\Delta \mathfrak{C} = \text{type}^\Delta \mathfrak{C} \cap R$.

Theorem 43. *Let R be an ordinary standard differential ring of finite differential type that satisfies **S3**. Suppose that either $\text{type}^\Delta R = 0$ or $\dim^\Delta R < \infty$, then R is a J-ring.*

Proof. Let $n > 0$ be a natural number. By Lemma 10, $1 \leq \text{type}^\Delta R^{\{n\}} < \infty$. First, we will show that $\text{type}^\Delta R^{\{n\}} = \max(\text{type}^\Delta R, 1)$.

If there exists such differential prime \mathfrak{q} in R such that $\text{ht}^\Delta \mathfrak{q}^{\{n\}} > 0$, then applying Remark 42 to the chain realizing differential height of \mathfrak{q} , we have $\text{type}^\Delta R \geq \text{type}^\Delta R^{\{n\}}$. Therefore, $\text{type}^\Delta R = \text{type}^\Delta R^{\{n\}}$.

In the second case, all extended primes have differential height zero. Suppose that $\text{type}^\Delta R = \text{type}^\Delta R^{\{n\}}$ then, by Remark 3, there exists $\mathfrak{q} \subset R$ of nonzero differential height, so $\text{ht}^\Delta \mathfrak{q}^{\{n\}} > 0$. Hence $\text{type}^\Delta R < \text{type}^\Delta R^{\{n\}}$.

By Remark 3, there exist $\mathfrak{p} \subset R^{\{n\}}$ of nonzero differential height. By Theorem 29, we have

$$0 < \text{ht}^\Delta \mathfrak{p} = \text{ht}^\Delta(\mathfrak{p} \cap R)^{\{n\}} + \text{ht}^\Delta \mathfrak{p}/(\mathfrak{p} \cap R)^{\{n\}} = \text{ht}^\Delta \mathfrak{p}/(\mathfrak{p} \cap R)^{\{n\}}.$$

But by Theorem 6, $\text{type}^\Delta \mathfrak{p}/\mathfrak{p} \cap R^{\{n\}} = 1$. Therefore, $\text{type}^\Delta R^{\{n\}} = 1 = \max(\text{type}^\Delta R, 1)$.

Now, we will prove the assertion about the differential dimension. If $\text{type}^\Delta R = 0$, then $\text{type}^\Delta R^{\{n\}} = 1$ and the property **S3** implies that, for every extended prime, $\text{ht}^\Delta \mathfrak{q}^{\{n\}} = 0$. By Theorem 29, $\text{ht}^\Delta \mathfrak{p} = \text{ht}^\Delta \mathfrak{p}/\mathfrak{p} \cap R^{\{n\}} \leq n$ for every differential prime \mathfrak{p} and, by Theorem 6, there exists \mathfrak{p} such that $\text{ht}^\Delta \mathfrak{p}/\mathfrak{p} \cap R^{\{n\}} = n$. So, $\dim^\Delta R^{\{n\}} = n$.

By Theorem 29, $\text{ht}^\Delta \mathfrak{p} = \text{ht}^\Delta(\mathfrak{p} \cap R)^{\{n\}} + \text{ht}^\Delta \mathfrak{p}/(\mathfrak{p} \cap R)^{\{n\}}$ for every differential prime \mathfrak{p} in $R^{\{n\}}$. Since **S3** holds, $\text{ht}^\Delta(\mathfrak{p} \cap R)^{\{n\}} = \text{ht}^\Delta \mathfrak{p} \cap R$. In the case of $\text{type}^\Delta R = 1$, we have $\text{ht}^\Delta \mathfrak{p} \leq \dim^\Delta R + n$. Consequently, $\dim^\Delta R^{\{n\}} = \dim^\Delta R + n$. Otherwise, if $\text{type}^\Delta R > 1$, we have $\text{ht}^\Delta \mathfrak{p} = \text{ht}^\Delta(\mathfrak{p} \cap R)^{\{n\}} \leq \dim^\Delta R$. Hence, $\dim^\Delta R^{\{n\}} = \dim^\Delta R$. \square

Remark 44. Since valuation domains of finite Krull dimension are Jaffard ([8, Corollary 2], [13, Theorem 4]), we could use Corollary 31 to prove that all Δ -arithmetical rings of finite Krull dimension are J-rings. However, we are able to prove this without the assumption to have finite Krull dimension.

Corollary 45. *Let R be an ordinary standard differential ring of finite differential type that has either differential type zero or finite differential dimension. If R is Δ -arithmetical then R is a J-ring.*

Proof. Let $n > 0$ be an integer, $\mathfrak{q}' \subset \mathfrak{q}$ differential primes in R and \mathfrak{p} be a differential upper to \mathfrak{q}' in $R^{\{n\}}$. An application of Proposition 37 to R/\mathfrak{q}' shows that R satisfies **S3**. By the previous theorem, R is J-ring. \square

Corollary 46. *Let R be an ordinary standard differential ring of finite differential type with locally nilpotent derivation that has either differential type zero or finite differential dimension. Then R is a J-ring.*

Proof. Let $n > 0$ be an integer, $\mathfrak{q}' \subset \mathfrak{q}$ differential primes in R and \mathfrak{p} be a differential upper to \mathfrak{q}' in $R^{\{n\}}$. Suppose that $\mathfrak{p} \subset \mathfrak{q}^{\{n\}}$. Let $f \in \mathfrak{p}R_{\mathfrak{q}}/\mathfrak{q}'R_{\mathfrak{q}}$, then $c \cdot f$ has ∂ -nilpotent coefficients, where c is the product of coefficients denominators of f . Since $R_{\mathfrak{q}}/\mathfrak{q}'R_{\mathfrak{q}}$ is a local integral domain, we have a contradiction with Lemma 41. Therefore, R satisfies **S3** and, consequently, is a J-ring. \square

We have shown that chains of differential prime ideals in polynomial extensions are affected by both the commutative structure of the base ring (Jaffard rings and Δ -arithmetical rings) and by properties of derivations (locally nilpotent derivation). This implies that standard rings that are not J-rings have to be rare. Namely, an example of a standard ordinary differential ring of finite Krull dimension that is not a J-ring should be a non-Jaffard ring that admits non-trivial derivation. Moreover, an example of a standard ordinary ring such that $\max(\text{type}^\Delta R, 1) < \text{type}^\Delta R\{z\}$ have to be even more complicated.

5. Acknowledgement

The author is very grateful to Evgeny Golod, who taught him commutative algebra and have acquainted him with the works of Gilmer and Seidenberg. The author is deeply appreciative to Alexander Levin for his advise and support. The author is very grateful to Dmitry Trushin, who taught him differential algebra and gave many crucial and useful suggestions including examples 1 and 2.

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